

Large deviations for multidimensional SDEs with reflection ^{*}

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Abstract

The large deviations principles are established for a class of multidimensional degenerate stochastic differential equations with reflecting boundary conditions. The results include two cases where the initial conditions are adapted and anticipated.

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1 Introduction and main results

Let \mathcal{O} be a smooth bounded open set in \mathbb{R}^d . $\mathbf{n}(x)$ denotes the cone of unit outward normal vectors to $\partial\mathcal{O}$ at x , that is,

$$(i) \quad \exists C_0 \geq 0, \forall x \in \partial\mathcal{O}, \forall x' \in \bar{\mathcal{O}}, \exists k \in \mathbf{n}(x) \\ \implies (x - x', k) + C_0|x - x'|^2 \geq 0, \quad (1.1)$$

$$(ii) \quad \forall x \in \partial\mathcal{O}, \text{ if } \exists C \geq 0, \exists k \in \mathbb{R}^d, \forall x' \in \bar{\mathcal{O}}, \\ (x - x', k) + C|x - x'|^2 \geq 0, \implies k = \theta \mathbf{n}(x) \quad (1.2) \\ \text{for some } \theta \geq 0. \text{ Moreover, we assume that}$$

$$(iii) \quad \exists n \geq 1, \exists \alpha > 0, \exists R > 0; \exists a_1, \dots, a_n \in \mathbb{R}^d, |a_i| = 1, \forall i, \\ \exists x_1, \dots, x_n \in \partial\mathcal{O} : \partial\mathcal{O} \subset \cup_{i=1}^n B(x_i, R), \\ \forall i, \forall x \in \partial\mathcal{O} \cap B(x_i, 2R), \forall \xi \in \mathbf{n}(x), (\xi, a_i) \geq \alpha > 0, \quad (1.3)$$

where $\partial\mathcal{O}$ denotes the boundary of \mathcal{O} , $\bar{\mathcal{O}}$ denotes the closure of \mathcal{O} , $B(x, r)$ denotes the ball of radius r at x . We assume that B_t is an \mathbb{R}^d -valued \mathcal{F}_t -Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,1]}, \mathbf{P})$ satisfying the

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usual assumptions. For $\varepsilon > 0$ we consider the following perturbed stochastic differential equations on domain \mathcal{O} with reflecting boundary conditions,

$$\begin{cases} X_t^\varepsilon(x) = x + \int_0^t b(X_s^\varepsilon(x))ds + \sqrt{\varepsilon} \int_0^t \sigma(X_s^\varepsilon(x))dB_s - L_t^\varepsilon(x), \\ L_t^\varepsilon(x) = \int_0^t \xi(X_s^\varepsilon(x))d|L^\varepsilon(x)|_s, \\ |L^\varepsilon(x)|_t = \int_0^t I_{\{s: X_s^\varepsilon(x) \in \partial\mathcal{O}\}} d|L^\varepsilon(x)|_s \end{cases} \quad (1.4)$$

for $\forall t \in [0, 1]$ and $x \in \bar{\mathcal{O}}$, where $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^d \times \mathbb{R}^d$ are continuous functions, $\xi(X_s^\varepsilon(x)) \in \mathbf{n}(X_s^\varepsilon(x))$, the $|L^\varepsilon(x)|_t$ denotes the total variation of $L_t^\varepsilon(x)$ on $[0, t]$. A pair $(X_t^\varepsilon(x), L_t^\varepsilon(x))$ of continuous processes is called a solution to equations (1.4) if there exists a measurable set $\tilde{\Omega}$ with $\mathbf{P}(\tilde{\Omega}) = 1$ such that for each $\omega \in \tilde{\Omega}$ (i) for each $x \in \bar{\mathcal{O}}$ the function $s \mapsto L_s^\varepsilon(x)$ with values in \mathbb{R}^d has bounded variation on interval $[0, 1]$ and $L_0^\varepsilon = 0$; (ii) for all $t \geq 0$, $X_t^\varepsilon(x) \in \bar{\mathcal{O}}$ and $(X_t^\varepsilon(x), L_t^\varepsilon(x))$ satisfies Eq.(1.4).

We first recall that the existence and uniqueness results on strong solutions to Eq.(1.4) were studied by Skorohod[11], Tanaka[12], Lions and Sznitman[8], Saisho[10], and other authors. When the initial value x is replaced by an arbitrary random variable $\{X_0^\varepsilon, \varepsilon > 0\}$, the author and Zhang (see [5, 6]) obtained recently existence of strong solution to Eq.(1.4), and we proved that the composition $(X_t^\varepsilon(X_0^\varepsilon), L_t^\varepsilon(X_0^\varepsilon))$ of stochastic processes $(X_t^\varepsilon(x), L_t^\varepsilon(x))$ and X_0^ε is just a solution of Eq.(1.4) corresponding to the initial X_0^ε under little regularity conditions on b , σ and shape of domain \mathcal{O} .

The goal of this paper will be two-fold. One is to establish the large deviation principle for $\{X_t^\varepsilon(x) : \varepsilon > 0\}$. We noted that this problem has been solved by Anderson and Orey [1], Cépa[2] and other authors. However, this result only deals with the case where the $\sigma\sigma^T$ is uniformly definite, i.e., σ is non-degenerate, because their proofs heavily depend on one dimensional stochastic differential equations with reflection in which the reflection has explicit representation formula. Here we will remove these restrictions and give detailed proof of the problem in case where σ is degenerate by a approach used by the author[7], which is somewhat different from that of [1, 2]. The other one is to prove that the non-adapted solution $\{X_t^\varepsilon(X_0^\varepsilon)\}$ also satisfies the large deviation principle under some hypotheses on the family $\{X_0^\varepsilon, \varepsilon > 0\}$ via some results on adapted solution $\{X_t^\varepsilon(x), \varepsilon > 0\}$.

To state our result more precisely, we introduce the following skeleton

equation associated with (1.4),

$$\begin{cases} z_t^\psi(x) = x + \int_0^t b(z_s^\psi(x))ds + \int_0^t \sigma(z_s^\psi(x))\psi(s)ds - k_t^\psi(x), \\ k_t^\psi(x) = \int_0^t \xi(z_s^\psi(x))d|k^\psi(x)|_s, \\ |k^\psi(x)|_t = \int_0^t I_{\{s: z_s^\psi(x) \in \partial\mathcal{O}\}}d|k^\psi(x)|_s \end{cases} \quad (1.5)$$

where $\psi \in L^2([0, 1]; \mathbb{R}^d)$, the $|k^\psi(x)|_t$ denotes the total variation of bounded variation function $k_t^\psi(x)$ on $[0, t]$ with $k_0^\psi(x) = 0$. A pair $(z_t^\psi(x), k_t^\psi(x))$ of functions is a solution of (1.5) means that $(z_t^\psi(x), k_t^\psi(x)) \in \bar{\mathcal{O}} \times \mathbb{R}^d$ satisfies Eq.(1.5).

Let $E = C([0, 1]; \bar{\mathcal{O}})$. $\|\cdot\|_E$ denotes the uniform norm on E . We define the rate functions I_1 , I_2^x and I^x by

$$I_1(g) = \begin{cases} \frac{1}{2} \int_0^1 |g(s)|^2 ds, & g \in L^2([0, 1]; \mathbb{R}^d), \\ +\infty, & \text{otherwise,} \end{cases} \quad (1.6)$$

$$I_2^x(f) = \inf\{I_1(\psi) : f = z^\psi(x) \text{ is a solution of Eq.(1.5)}\}, \quad (1.7)$$

$$I^x(f) = \limsup_{y \rightarrow x} \{I_2^y(f)\} \quad (1.8)$$

for $x \in \bar{\mathcal{O}}$ and $f \in E$. We set $\inf \emptyset = +\infty$ by convention. The main results of this paper are the following.

Theorem 1.1. *Assume that \mathcal{O} is a smooth bounded open set in \mathbb{R}^d satisfying (1.1)-(1.3) and there exists a function $\phi \in C_b^2(\mathbb{R}^d)$ such that*

$$\exists \alpha > 0, \forall x \in \partial\mathcal{O}, \forall \zeta \in \mathbf{n}(x), (\nabla\phi(x), \zeta) \leq -\alpha C_0. \quad (1.9)$$

Let b and σ be bounded and uniformly Lipschitz. $\{X_t^\varepsilon(x), \varepsilon > 0\}$ is the uniqueness solution of Eq.(1.4). Then $\{X^\varepsilon(x), \varepsilon > 0\}$ satisfies large deviation principle on E with good rate function I_2^x given by (1.7). In other words, for any open set G and any closed set F of E , we have

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\{X^\varepsilon(x) \in G\} \geq - \inf_{f \in G} \{I_2^x(f)\}, \quad (1.10)$$

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\{X^\varepsilon(x) \in F\} \leq - \inf_{f \in F} \{I_2^x(f)\}. \quad (1.11)$$

Theorem 1.2. *Assume the conditions of Theorem 1.1. σ and b are bounded and satisfy the following*

$$\begin{aligned} & |b(x) - b(y)| + \|\sigma(x) - \sigma(y)\| + \|(\nabla\sigma \cdot \sigma)(x) - (\nabla\sigma \cdot \sigma)(y)\| \\ & \|(\nabla\sigma \cdot \nabla\sigma \cdot \sigma)(x) - (\nabla\sigma \cdot \nabla\sigma \cdot \sigma)(y)\| + \|(\nabla\sigma \cdot b)(x) - (\nabla\sigma \cdot b)(y)\| \\ & + \|(\sigma^T \cdot \nabla^2\sigma \cdot \sigma)(x) - (\sigma^T \cdot \nabla^2\sigma \cdot \sigma)(y)\| \leq C|x - y| \end{aligned} \quad (1.12)$$

for some constants $C > 0$, where σ^T denotes transpose of σ , $\nabla\sigma$ and $\nabla^2\sigma$ denote σ 's derivatives of first and second order with respect to spatial variable x , respectively. Then for any random variable X_0^ε with $\mathbf{P}\{X_0^\varepsilon \in \bar{\mathcal{O}}\} = 1$ and the family $\{X_0^\varepsilon, \varepsilon > 0\}$ satisfies for $x_0 \in \bar{\mathcal{O}}$ and any $\delta > 0$

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\{|X_0^\varepsilon - x_0| > \delta\} = -\infty, \quad (1.13)$$

the processes $\{X_t^\varepsilon(X_0^\varepsilon) : \varepsilon > 0\}$ satisfy the large deviation principle on E with good rate function I^{x_0} given by (1.8). In other words, for any open set G and any closed set F of E , we have

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\{X^\varepsilon(X_0^\varepsilon) \in G\} \geq -\inf_{f \in G} \{I^{x_0}(f)\}, \quad (1.14)$$

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\{X^\varepsilon(X_0^\varepsilon) \in F\} \leq -\inf_{f \in F} \{I^{x_0}(f)\}. \quad (1.15)$$

This paper is organized as follows. We will study the skeleton equation (1.5) and its Euler approximation in section 2; In section 3 we will give an exponential approximation of the solution $\{X_t^\varepsilon\}_{t \in [0,1]}$. In sections 4 and 5, we will prove Theorems 1.1 and 1.2.

Throughout this paper we make the following convention: the letter c or $c(p_1, p_2, p_3, \dots)$ depending only on p_1, p_2, p_3, \dots will denote an unimportant positive constant, whose values may change from one line to another one.

2 Skeleton Equations (1.5)

In this section we mainly study existence, uniqueness and approximation of solution of the skeleton equations (1.5). The first result is the following.

Theorem 2.1. *Assume that \mathcal{O} is a smooth bounded open set in \mathbb{R}^d satisfying (1.1)-(1.2) and (1.9). Let b and σ be bounded and uniformly Lipschitz. Then for any $x \in \bar{\mathcal{O}}$ and any $\psi \in L^2([0,1]; \mathbb{R}^d)$ the skeleton equation (1.5) has a unique solution $(z_t^\psi(x), k_t^\psi(x))$ and*

$$\|z^{\psi_1} - z^{\psi_2}\|_t^2 \leq c(r) \int_0^t \|\psi_1 - \psi_2\|_s^2 ds \quad (2.1)$$

holds for some constants $c(r)$, $\psi_1, \psi_2 \in L^2([0,1]; \mathbb{R}^d)$ with $\|\psi_1\|_{L^2}^2 \leq r$ and $\|\psi_2\|_{L^2}^2 \leq r$, where $\|f\|_t := \sup_{s \in [0,t]} \{|f(s)|\}$ and $r > 0$.

Proof. For $\psi \in L^2([0,1]; \mathbb{R}^d)$, define $F(\cdot) : H := \{f \in E_1; \|f\|_{E_1} < +\infty\} \rightarrow \mathbb{R}^d$ by

$$\begin{cases} F(z_t) = x + \int_0^t b(z_s) ds + \int_0^t \sigma(z_s) \psi(s) ds - k_t^\psi(x), \\ k_t^\psi(x) = \int_0^t \xi(F(z_s)) d|k^\psi(x)|_s, \\ |k^\psi(x)|_t = \int_0^t I_{\{s: F(z_s) \in \partial\mathcal{O}\}} d|k^\psi(x)|_s, \end{cases} \quad (2.2)$$

where $E_1 = C([0, 1]; \mathbb{R}^d)$. Then there exists a constant $c = c(\psi) > 0$ such that $\forall t \in [0, 1], \forall z, z' \in H$ we have

$$\|X - X'\|_t^4 \leq c \int_0^t \|z - z'\|_s^4 ds, \quad (2.3)$$

where $X = F(z)$ and $X' = F(z')$.

We first prove (2.3).

By chain rule, we have for $\phi \in C_b^2(\bar{O})$

$$\begin{aligned} \phi(X_t) &= \phi(x) + \int_0^t \nabla \phi(X_s) \cdot b(z_s) ds + \int_0^t \nabla \phi(X_s) \cdot \sigma(z_s) \psi(s) ds \\ &\quad - \int_0^t \nabla \phi(X_s) \cdot \xi(X_s) d|k^\psi(x)|_s, \end{aligned} \quad (2.4)$$

$$\begin{aligned} \phi(X'_t) &= \phi(x) + \int_0^t \nabla \phi(X'_s) \cdot b(z'_s) ds + \int_0^t \nabla \phi(X'_s) \cdot \sigma(z'_s) \psi(s) ds \\ &\quad - \int_0^t \nabla \phi(X'_s) \cdot \xi(X'_s) d|k'^\psi(x)|_s. \end{aligned} \quad (2.5)$$

Using conditions (1.1), (1.2) and (1.9) on O ,

$$\frac{1}{\alpha} (\nabla \phi(z_s), \xi(z_s)) |z_s - z'_s|^2 - (z_s - z'_s, \xi(z_s)) \leq 0, \quad d|k^\psi(x)|_s, \text{ a.s.}, \quad (2.6)$$

$$\frac{1}{\alpha} (\nabla \phi(z'_s), \xi(z'_s)) |z_s - z'_s|^2 - (z_s - z'_s, \xi(z'_s)) \leq 0, \quad d|k'^\psi(x)|_s, \text{ a.s.}. \quad (2.7)$$

We now calculate the following term:

$$\exp \left\{ -\frac{2}{\alpha} [\phi(X_t) + \phi(X'_t)] \right\} |X_t - X'_t|^2.$$

Let $f(t) = \exp \left\{ -\frac{2}{\alpha} [\phi(X_t) + \phi(X'_t)] \right\}$. By chain rule we have

$$\begin{aligned} f(t) |X_t - X'_t|^2 &= 2 \int_0^t f(s) (X_s - X'_s) \cdot (b(z_s) - b(z'_s)) ds \\ &\quad + 2 \int_0^t f(s) (X_s - X'_s) \cdot (\sigma(z_s) - \sigma(z'_s)) \cdot \psi(s) ds \\ &\quad - \frac{2}{\alpha} \int_0^t f(s) |X_s - X'_s|^2 [\nabla \phi(X_s) \cdot b(z_s) + \nabla \phi(X'_s) \cdot b(z'_s) \\ &\quad + \nabla \phi(X_s) \cdot \sigma(z_s) \psi(s) + \nabla \phi(X'_s) \cdot \sigma(z'_s) \psi(s)] ds \\ &\quad + 2 \int_0^t f(s) \left[\frac{1}{\alpha} (\nabla \phi(z_s), \xi(z_s)) |z_s - z'_s|^2 - (z_s - z'_s, \xi(z_s)) \right] d|k^\psi(x)|_s \\ &\quad + 2 \int_0^t f(s) \left[\frac{1}{\alpha} (\nabla \phi(z'_s), \xi(z'_s)) |z_s - z'_s|^2 - (z_s - z'_s, \xi(z'_s)) \right] d|k'^\psi(x)|_s. \end{aligned} \quad (2.8)$$

Since b and σ are bounded and uniformly Lipschitz, $\phi \in C_b^2(\bar{\mathcal{O}})$, we deduce from (2.6)-(2.8) that there exists a constant $c > 0$ such that

$$\begin{aligned} |X_t - X'_t|^2 &\leq c \int_0^t |X_s - X'_s| |z_s - z'_s| [1 + |\psi(s)|] ds \\ &\quad + c \int_0^t |X_s - X'_s|^2 ds. \end{aligned} \quad (2.9)$$

Consequently,

$$\begin{aligned} |X_t - X'_t|^4 &\leq c(1 + \|\psi\|_{L^2}^2) \int_0^t |X_s - X'_s|^4 ds \\ &\quad + c(1 + \|\psi\|_{L^2}^2) \int_0^t |z_s - z'_s|^4 ds. \end{aligned} \quad (2.10)$$

Letting $c_1 = c(1 + \|\psi\|_{L^2}^2) \exp\{c(1 + \|\psi\|_{L^2}^2)\}$, by the Gronwall lemma, we obtain that

$$|X_t - X'_t|^4 \leq c_1 \int_0^t |z_s - z'_s|^4 ds. \quad (2.11)$$

So the proof of (2.3) has been done.

Next we turn to proving existence of solution of (1.5).

Define a sequence $\{Y_n(s)\}_{n=1}^\infty$ of functions by

$$\begin{cases} Y_0(t) = x, \\ Y_n(t) = F(Y_{n-1}(t)), \quad t \in [0, 1]. \end{cases} \quad (2.12)$$

By (2.3), for large enough n and some constants $c > 0$, we have

$$\|Y_n - Y_{n-1}\|_{E_1} \leq \frac{c}{n^2}.$$

Define $Y(t)$ on $[0, 1]$ by

$$Y(t) = Y_0(t) + \sum_{n=1}^{\infty} (Y_n(t) - Y_{n-1}(t)).$$

Then

$$\|Y_n - Y\|_{E_1} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Using the first equation in (2.2) and (2.12), there exists a $k \in C([0, 1]; \mathbb{R}^d) \cap BV([0, 1])$ ($BV([0, 1])$ denotes the space of bounded variation functions on $[0, 1]$) such that

$$\|k_n - k\|_{E_1} \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

where k_n is the local time of Y_n . So we can assume that (Y_n, k_n) converges uniformly to (Y, k) , which are continuous on $[0, 1]$. Letting $n \rightarrow +\infty$ in

(2.12) we know that (Y, k) solves the first equation in (1.5). The remains of proving existence is to check the second and third equations in (1.5). Let $f \in C([0, 1]; \mathbb{R}^d)$, $0 \leq f \leq 1$, $f|_A = 1$, A being any compact subset of O . By Fatou lemma,

$$0 \leq \int_0^t f(Y_s) d|k|_s \leq \liminf_{n \rightarrow +\infty} \int_0^t f(Y_n(s)) d|k_n|_s = 0,$$

which implies that

$$\int_0^t I_{\{s: Y_s \in \partial O\}} d|k|_s = 0.$$

Thus (Y, k) solves the second equation in (1.5).

Using conditions (1.1), (1.2) and (1.9) on O again, for any $f \in C([0, 1]; \mathbb{R})$ with $f \geq 0$ and any $\beta \in \bar{O}$, we have

$$\int_0^t f(s)(Y_n(s) - \beta, dk_n(s)) + C_0 \int_0^t |Y_n(s) - \beta|^2 f(s) d|k_n|_s \geq 0.$$

Noticing that the measure $d|k_n|$ converges to some measure da_s as $n \rightarrow +\infty$, we get that

$$\int_0^t f(s)(Y(s) - \beta, dk(s)) + C_0 \int_0^t |Y(s) - \beta|^2 f(s) da_s \geq 0.$$

Since $d|k|_s \leq da_s$, we have $dk_s = h_s da_s$, h_s being a bounded measurable function, thus

$$(Y(s) - \beta, h_s) + C_0 |Y(s) - \beta|^2 \geq 0,$$

which, together with (1.1) and (1.2), implies that $h_s \in \lambda \mathbf{n}(Y_s)$ for some $\lambda \geq 0$. Hence, we find a $\xi(Y_s) \in \mathbf{n}(Y_s)$ such that

$$k_t = \int_0^t \xi(Y_s) d|k|_s,$$

that is, (Y, k) also solves the third equation in (1.5). Therefore we have proved the existence of solution of (1.5).

Let (Y, k) and (Y', k') be two solutions of (1.5) with $Y(0) = Y'(0) = x$ and $k_0 = k'_0 = 0$. By (2.3),

$$\|Y - Y'\|_{E_1}^4 \leq c \int_0^1 \|Y - Y'\|_s^4 ds.$$

So $Y = Y'$. We then deduce from the first equation in (1.5) that $k = k'$. This implies the uniqueness of solution of Eq.(1.5).

To the proof of Theorem 2.1 end, we to prove (2.1). Let $\psi_i \in L^2([0, 1]; \mathbb{R}^d)$ and $(z_t^{\psi_i}(x), k_t^{\psi_i}(x))$ be the solutions of Eq.(1.5) corresponding to ψ_i for

$i = 1, 2$. Replacing $f(t)$ by $\exp \left\{ -\frac{2}{\alpha} [\psi_1(z_t^{\psi_1}(x)) + \psi_2(z_t^{\psi_2}(x))] \right\}$ in (2.8) and using the same way as in (2.10), we can prove that for any $r > 0$ and any $\|\psi_i\|_{L^2}^2 \leq r$ for $i = 1, 2$,

$$\begin{aligned} |z_t^{\psi_1}(x) - z_t^{\psi_2}(x)|^2 &\leq c(r) \int_0^t |z_s^{\psi_1}(x) - z_s^{\psi_2}(x)|^2 \\ &\quad + c(r) \int_0^t |\psi_1(s) - \psi_2(s)|^2 ds. \end{aligned} \quad (2.13)$$

Thus by Gronwall lemma we complete the proof. \square

For $n \geq 1$, let ϕ_n be a step function on $[0, 1]$ defined by $\phi_n(t) = \frac{k}{2^n}$ for $t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]$, $k = 0, 1, 2, \dots, 2^n - 1$. Let $\{z_n^\psi\} \in C([0, 1] \times \bar{\mathcal{O}}; \bar{\mathcal{O}})$ be the Euler approximating sequence of Eq.(1.5) defined by the following

$$\begin{cases} z_n^\psi(t, x) = x + \int_0^t b(z_n^\psi(\phi_n(s), x)) ds + \int_0^t \sigma(z_n^\psi(\phi_n(s), x)) \psi(s) ds - k_n^\psi(t), \\ k_n^\psi(t) = \int_0^t \xi(z_n^\psi(s, x)) d|k_n^\psi|_s, \\ |k_n^\psi|_t = \int_0^t I_{\{s: z_n^\psi(s, x) \in \partial \mathcal{O}\}} d|k_n^\psi|_s \end{cases} \quad (2.14)$$

for $n \geq 1$ and $\psi \in L^2([0, 1]; \mathbb{R}^d)$. The second result of this section is the following.

Theorem 2.2. *Let \mathcal{O} satisfy conditions (1.1)-(1.3). Then for any $r > 0$ we have*

$$\lim_{n \rightarrow \infty} \sup_{\{\psi: I_1(\psi) \leq r\}} \|z_n^\psi - z^\psi\|_{E_2} = 0, \quad (2.15)$$

where $\|\cdot\|_{E_2}$ denotes the uniform norm of $E_2 := C([0, 1] \times \bar{\mathcal{O}}; \bar{\mathcal{O}})$.

Proof. By (1.5) and (2.14)

$$\begin{aligned} z_n^\psi(t, x) - z_t^\psi(x) &= \int_0^t [b(z_n^\psi(\phi_n(s), x)) - b(z_s^\psi(x))] ds \\ &\quad + \int_0^t [\sigma(z_n^\psi(\phi_n(s), x)) - \sigma(z_s^\psi(x))] \psi(s) ds \\ &\quad - [k_n^\psi(t) - k^\psi(s)] := \omega(t) - L_n^\psi(t). \end{aligned} \quad (2.16)$$

Since $I_1(\psi) \leq r$, b and σ are bounded, we know that $\omega \in BV([0, 1])$. Using Theorem 2.1 in [8], we get that

$$\begin{aligned} |L_n^\psi|_t &\leq \int_0^t |b(z_n^\psi(\phi_n(s), x)) - b(z_s^\psi(x))| ds \\ &\quad + \int_0^t |[\sigma(z_n^\psi(\phi_n(s), x)) - \sigma(z_s^\psi(x))] \psi(s)| ds. \end{aligned} \quad (2.17)$$

So we deduce from (2.16) that

$$\begin{aligned}
|z_n^\psi(t, x) - z_t^\psi(x)| &\leq 2 \int_0^t |b(z_n^\psi(\phi_n(s), x)) - b(z_s^\psi(x))| ds \\
&+ 2 \int_0^t |\sigma(z_n^\psi(\phi_n(s), x)) - \sigma(z_s^\psi(x))| |\psi(s)| ds.
\end{aligned} \tag{2.18}$$

Similarly,

$$\begin{aligned}
|z_n^\psi(t, x) - z_n^\psi(\phi_n(t), x)| &\leq 2 \int_{\phi_n(t)}^t |b(z_n^\psi(\phi_n(s), x))| ds \\
&+ 2 \int_{\phi_n(t)}^t |\sigma(z_n^\psi(\phi_n(s), x))| |\psi(s)| ds. \\
&\leq \frac{2c}{2^n} [1 + \int_0^1 |\psi(s)|] ds \\
&\leq c(r) \frac{1}{2^n}.
\end{aligned} \tag{2.19}$$

Hence, using (2.18) and (2.19) as well as b and σ are uniformly Lipschitz we have

$$\begin{aligned}
&\sup_{\{x \in \bar{\mathcal{O}}, s \leq t\}} \{|z_n^\psi(s, x) - z_s^\psi(x)|\} \\
&\leq c \int_0^t [1 + |\psi(s)|] [\sup_{\{x \in \bar{\mathcal{O}}, u \leq s\}} \{|z_n^\psi(u, x) - z_u^\psi(x)|\}] ds \\
&+ c(r) \frac{1}{2^n} \int_0^t [1 + |\psi(s)|] ds,
\end{aligned} \tag{2.20}$$

which, together with Gronwall lemma, implies that

$$\|z_n^\psi - z^\psi\|_{E_2} \leq c(r) \frac{1}{2^n} \int_0^1 [1 + |\psi(s)|] ds \exp\{c \int_0^1 [1 + |\psi(s)|] ds\}. \tag{2.21}$$

Since $\int_0^1 [1 + |\psi(s)|] ds \leq (1 + \sqrt{2r})$, letting $n \rightarrow \infty$ in (2.21) we have

$$\lim_{n \rightarrow \infty} \sup_{\{\psi: I_1(\psi) \leq r\}} \|z_n^\psi - z^\psi\|_{E_2} = 0.$$

Thus we complete the proof of Theorem 2.2. \square

We define $F_n(\cdot)$ from E_1 to E_2 by

$$\left\{ \begin{array}{l} F_n(\omega)(0, x) = x, \\ F_n(\omega)(t, x) = F_n(\omega)(\frac{k}{2^n}, x) + b(F_n(\omega)(\frac{k}{2^n}, x))(t - \frac{1}{2^n}) \\ \quad + \sigma(F_n(\omega)(\frac{k}{2^n}, x))(\omega(t) - \omega(\frac{1}{2^n})) \\ \quad - \int_{\frac{k}{2^n}}^t \xi(F_n(\omega)(s, x)) d|L_n^x|_s, \\ L_n^x(s) = \int_0^t \xi(F_n(\omega)(s, x)) d|L_n^x|_s, \\ |L_n^x|_t = \int_0^t I_{\{s: F_n(\omega)(s, x) \in \partial \mathcal{O}\}} d|L_n^x|_s \end{array} \right. \tag{2.22}$$

for $n \geq 1$ and $k = 0, 1, \dots, 2^n - 1$. The third result of this section is the following.

Theorem 2.3. *Let O satisfy conditions (1.1)-(1.3) and (1.9). Then for $n \geq 1$ the maps $F_n(\cdot)$ from E_1 to E_2 are continuous.*

Before proving proof of Theorem 2.3, we need the following lemmas.

Lemma 2.1. *Let the smooth bounded O satisfy (1.1)-(1.3) and $w \in E$. Assume that for every $x \in \bar{O}$, (Y_x, k_x) is the unique solution of the following Skorohod equation(abbreviated by $(\omega_x, O, \mathbf{n})$),*

$$\begin{cases} Y_x(t) = \omega(t, x) - k_x(t), \\ k_x(t) = \int_0^t \xi(Y_x(s)) d|k_x|_s, \\ |k_x|_t = \int_0^t I_{\{s: Y_x(s) \in \partial O\}} d|k_x|_s. \end{cases} \quad (2.23)$$

Then

$$\|k\|_{E_2} \leq M(\alpha, C_0, \bar{O}) < +\infty, \quad (2.24)$$

where $M(\alpha, C_0, \bar{O})$ are some positive constants depending only on α , C_0 and \bar{O} .

Proof. Since \bar{O} satisfies (1.3) and is bounded, the proof is completely similar to that of Lemma 1.2 in [8], we omit it here. \square

Lemma 2.2. *Let the smooth bounded O satisfy (1.1) and (1.2). Assume that (Y_i, k_i) , $i = 1, 2$, are unique solutions of Skorohod equations (w_1, O, \mathbf{n}) and (w_2, O, \mathbf{n}) , respectively. Then*

$$\begin{aligned} \|Y_1 - Y_2\|_t^2 &\leq [1 + \exp\{2C_0(|k_1|_t + |k_2|_t)\}] \|\omega_1 - \omega_2\|_t^2 \\ &\quad + \frac{3}{C_0} \exp\{4C_0(|k_1|_t + |k_2|_t)\} \|\omega_1 - \omega_2\|_t. \end{aligned} \quad (2.25)$$

Proof. Since the first inequality (6) in Lemma 1.1 in [8] holds for $\omega \in E$ due to constant C there can be replaced by $2C_0$, we have

$$\begin{aligned} |Y_1(t) - Y_2(t)|^2 &\leq |\omega_1(t) - \omega_2(t)|^2 + 2(k_1(t) - k_2(t)) \cdot (\omega_1(t) - \omega_2(t)) \\ &\quad + \exp\{2C_0(|k_1|_t + |k_2|_t)\} \cdot \{\|\omega_1 - \omega_2\|_t^2 \\ &\quad + \frac{1}{C_0} \|\omega_1 - \omega_2\|_t + \|(\omega_1 - \omega_2) \cdot (k_1 - k_2)\|_t\}, \end{aligned} \quad (2.26)$$

which implies that (2.25). \square

Now we turn to proving Theorem 2.3.

Proof of Theorem 2.3. For $\omega_i \in E_1$ with $\|\omega_i\|_{E_1} < +\infty$, $i = 1, 2$, define $(F_n(\omega_i), L_n^{x,i})$ by (2.22). Let

$$\begin{aligned}\widetilde{w}_i(t, x) &= F_n(\omega_i)\left(\frac{k}{2^n}, x\right) + b(F_n(\omega_i)\left(\frac{k}{2^n}, x\right))\left(t - \frac{1}{2^n}\right) \\ &\quad + \sigma(F_n(\omega_i)\left(\frac{k}{2^n}, x\right))\left(\omega_i(t) - \omega_i\left(\frac{1}{2^n}\right)\right).\end{aligned}\quad (2.27)$$

Then

$$\begin{cases} F_n(\omega_i)(t, x) = \widetilde{w}_i(t, x) - L_n^{x,i}, \\ L_n^{x,i}(s) = \int_0^t \xi(F_n(\omega_i)(s, x)) d|L_n^{x,i}|_s, \\ |L_n^{x,i}|_t = \int_0^t I_{\{s: F_n(\omega_i)(s, x) \in \partial\mathcal{O}\}} d|L_n^{x,i}|_s. \end{cases}\quad (2.28)$$

By Lemma 2.1, $\exists M_1 = M_1(\alpha, O, C_0) < +\infty$ such that for $i = 1, 2$,

$$\sup_{n \geq 1} \{\|L_n^{x,i}\|_{E_2}\} \leq M_1. \quad (2.29)$$

Setting $Y_n(t, x) = F_n(\omega_1)(t, x) - F_n(\omega_2)(t, x)$. For $t \in [0, \frac{1}{2^n}]$, by Lemma 2.2 and (2.4), $\exists M_2 = M_2(\alpha, O, C_0) > 0$ such that

$$\|Y_n(\cdot, x)\|_{\frac{1}{2^n}} \leq M_2 \left\{ \|\widetilde{w}_1(\cdot, x) - \widetilde{w}_2(\cdot, x)\|_{\frac{1}{2}}^2 + \|\widetilde{w}_1(\cdot, x) - \widetilde{w}_2(\cdot, x)\|_{\frac{1}{2}} \right\}^{\frac{1}{2}}, \quad (2.30)$$

where $\|f\|_t = \sup_{s \leq t} \{|f(s)|\}$. Noticing that for $t \in [0, \frac{1}{2^n}]$,

$$\widetilde{w}_1(\cdot, x) - \widetilde{w}_2(\cdot, x) = \sigma(x)[(\omega_1(t) - \omega_2(t)) - (\omega_1(0) - \omega_2(0))],$$

we have

$$\|\widetilde{w}_1(\cdot, x) - \widetilde{w}_2(\cdot, x)\|_{\frac{1}{2^n}} \leq c \|\omega_1 - \omega_2\|_{\frac{1}{2^n}}. \quad (2.31)$$

Using (2.30) and (2.31), we have

$$\|Y_n(\cdot, x)\|_{\frac{1}{2^n}} \leq c(\alpha, O, C_0) \left\{ \|\omega_1 - \omega_2\|_{\frac{1}{2}}^2 + \|\omega_1 - \omega_2\|_{\frac{1}{2}} \right\}^{\frac{1}{2}}. \quad (2.32)$$

Since

$$\begin{aligned}\|Y_n(\cdot, x)\|_{\frac{2}{2^n}} &\leq \|Y_n(\cdot, x)\|_{\frac{1}{2^n}} \\ &\quad + \sup_{t \in \Delta_2} \left\{ \left| \int_{\frac{1}{2^n}}^t [b(F_n(\omega_1)\left(\frac{k}{2^n}, x\right)) - b(F_n(\omega_2)\left(\frac{k}{2^n}, x\right))] ds \right| \right\} \\ &\quad + \sup_{t \in \Delta_2} \left\{ \left| \left[\sigma(F_n(\omega_1)\left(\frac{k}{2^n}, x\right)) - \sigma(F_n(\omega_2)\left(\frac{k}{2^n}, x\right)) \right] (\omega_1(t) - \omega_1(0)) \right| \right\} \\ &\quad + \sup_{t \in \Delta_2} \left\{ \left| \sigma(F_n(\omega_2)\left(\frac{k}{2^n}, x\right)) [(\omega_1(t) - \omega_2(t)) - (\omega_1(0) - \omega_2(0))] \right| \right\},\end{aligned}\quad (2.33)$$

where $\Delta_2 := [\frac{1}{2^n}, \frac{2}{2^n}]$, we have

$$\|Y_n(\cdot, x)\|_{\frac{2}{2^n}} \leq c(\alpha, O, C_0, \|\omega_1\|_{E_1}) \|\omega_1 - \omega_2\|_{E_1}^{\frac{1}{2}} (1 + \|\omega_1 - \omega_2\|_{E_1}^{\frac{1}{2}}) + c\|\omega_1 - \omega_2\|_{E_1}.$$

Doing the same procedure as in estimating $\|Y_n(\cdot, x)\|_{\frac{2}{2^n}}$, we can find $c_1 = c_1(\alpha, O, C_0, \|\omega_1\|_{E_1})$ and $c_2 = c_2(\alpha, O, C_0, \|\omega_1\|_{E_1})$ such that

$$\|Y_n(\cdot, \cdot)\|_{E_2} \leq c_1 \|\omega_1 - \omega_2\|_{E_1}^{\frac{1}{2}} (1 + \|\omega_1 - \omega_2\|_{E_1}^{\frac{1}{2}}) + c_2 \|\omega_1 - \omega_2\|_{E_1}.$$

Thus we complete the proof by the last inequality. \square

3 Exponential approximation of the solution for SDE(1.4)

We consider the following Euler approximation of SDE (1.4),

$$\begin{cases} X_n^\varepsilon(t, x) = x + \int_0^t b(X_n^\varepsilon(\phi_n(s), x))ds + \sqrt{\varepsilon} \int_0^t \sigma(X_n^\varepsilon(\phi_n(s), x))dB_s \\ \quad - L_n^\varepsilon(t, x), \\ L_n^\varepsilon(t, x) = \int_0^t \xi(X_n^\varepsilon(s, x))d|L_n^\varepsilon(x)|_s, \\ |L_n^\varepsilon(x)|_t = \int_0^t I_{\{s: X_n^\varepsilon(s, x) \in \partial\mathcal{O}\}} d|L_n^\varepsilon(x)|_s \end{cases} \quad (3.1)$$

for $n \geq 1$ and $x \in \bar{\mathcal{O}}$. The main result of this section is the following.

Theorem 3.1. *Let O satisfy conditions(1.1)-(1.3) and (1.9). Then we have*

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P} \left\{ \sup_{t \in [0, 1]} |X_t^\varepsilon(x) - X_n^\varepsilon(t, x)| \geq \delta \right\} = -\infty \quad (3.2)$$

for any $\delta > 0$.

Proof. We first define stopping time τ_1 for $\delta_1 > 0$ by

$$\tau_1 = \inf \{t \geq 0 : |X_n^\varepsilon(t) - X_n^\varepsilon(\phi_n(t))| \geq \delta_1\} \wedge 1.$$

Then

$$\mathbf{P}\{\tau_1 \leq 1\} \leq \sum_{k=0}^{2^n-1} \mathbf{P}\left\{ \sup_{t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]} |X_n^\varepsilon(t) - X_n^\varepsilon(\phi_n(t))| \geq \delta_1 \right\}. \quad (3.3)$$

Since

$$\begin{aligned} X_n^\varepsilon(t) - X_n^\varepsilon(\phi_n(t)) &= \int_{\phi_n(t)}^t b(X_n^\varepsilon(\phi_n(s), x))ds + \sqrt{\varepsilon} \int_{\phi_n(t)}^t \sigma(X_n^\varepsilon(\phi_n(s), x))dB_s \\ &\quad - \int_{\phi_n(t)}^t \xi(X_n^\varepsilon(s, x))d|L_n^\varepsilon(x)|_s, \end{aligned} \quad (3.4)$$

by using (2.24) and (2.25), there exists a positive constant c_1 depending only on α , O and C_0 such that

$$\begin{aligned}
& |X_n^\varepsilon(t) - X_n^\varepsilon(\phi_n(t))| \\
& \leq c \left[\left| \int_{\phi_n(t)}^t b(X_n^\varepsilon(\phi_n(s), x)) ds + \sqrt{\varepsilon} \int_{\phi_n(t)}^t \sigma(X_n^\varepsilon(\phi_n(s), x)) dB_s \right| \right. \\
& \quad \left. + \left| \int_{\phi_n(t)}^t b(X_n^\varepsilon(\phi_n(s), x)) ds + \sqrt{\varepsilon} \int_{\phi_n(t)}^t \sigma(X_n^\varepsilon(\phi_n(s), x)) dB_s \right|^{\frac{1}{2}} \right] \\
& \leq \left[\frac{c_2}{2^n} + c_2 \sqrt{\varepsilon} \max_{0 \leq t \leq \frac{1}{2^n}} \{|\tilde{B}_t|\} \right] + \sqrt{\frac{c_2}{2^n} + c_2 \sqrt{\varepsilon} \max_{0 \leq t \leq \frac{1}{2^n}} \{|\tilde{B}_t|\}} \\
& := b_n + \sqrt{b_n},
\end{aligned} \tag{3.5}$$

where $\tilde{B}_t = B(t + \frac{k}{2^n} - B(\frac{k}{2^n}))$ is also a Brownian motion. So, by choosing $\frac{\delta_1}{2} \leq 1$, we have

$$\begin{aligned}
& \mathbf{P} \left\{ \sup_{t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]} |X_n^\varepsilon(t) - X_n^\varepsilon(\phi_n(t))| \geq \delta_1 \right\} \\
& \leq \mathbf{P} \left\{ b_n \geq \frac{\delta_1}{2} \right\} + \mathbf{P} \left\{ b_n \geq \left(\frac{\delta_1}{2} \right)^2 \right\} \\
& \leq 2\mathbf{P} \left\{ b_n \geq \left(\frac{\delta_1}{2} \right)^2 \right\} \\
& \leq 2\mathbf{P} \left\{ \max_{0 \leq t \leq \frac{1}{2^n}} \{|\tilde{B}_t|\} \geq \frac{(\frac{\delta_1^2}{4} - \frac{c_1}{2^n})}{\sqrt{\varepsilon} c_1} \right\} \\
& \leq 8d \exp \left\{ - \frac{2^n (\frac{\delta_1^2}{4} - \frac{c_1}{2^n})^2}{2d\varepsilon c_1^2} \right\}
\end{aligned} \tag{3.6}$$

by Lemma 5.2.1 in [3]. So

$$\mathbf{P} \{ \tau_1 \leq 1 \} \leq 2^n 8d \exp \left\{ - \frac{2^n (\frac{\delta_1^2}{4} - \frac{c_1}{2^n})^2}{2d\varepsilon c_1^2} \right\}.$$

Consequently,

$$\varepsilon \log \mathbf{P} \{ \tau_1 \leq 1 \} \leq \varepsilon n \log 2 + \varepsilon \log(8d) - 2^n \frac{(\frac{\delta_1^2}{4} - \frac{c_1}{2^n})^2}{2d\varepsilon c_1^2}.$$

Hence,

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P} \{ \tau_1 \leq 1 \} = -\infty. \tag{3.7}$$

Next we define functions Ψ and Φ_λ on $[0, +\infty)$ by

$$\Psi(x) = \int_0^x \frac{ds}{\delta_1^2 + s} \quad \text{and} \quad \Phi_\lambda(x) = \exp \{ \lambda \Psi(x) \} \quad \text{for } \lambda > 0.$$

For $\delta > 0$, define stopping time τ_2 by

$$\tau_2 = \inf\{t > 0 : |z_t| \geq \delta\} \wedge \tau_1,$$

where

$$z_t = X_t^\varepsilon(x) - X_n^\varepsilon(t, x).$$

Let

$$m_t = z_{t \wedge \tau_2}, \quad f(x) = |x|^2, \quad x \in \mathbb{R}^d, \quad (3.8)$$

$$\tilde{b}_t = b(X_{t \wedge \tau_2}^\varepsilon(x)) - b(X_n^\varepsilon(\phi_n(t), x)), \quad (3.9)$$

$$\tilde{\sigma}_t = \sqrt{\varepsilon}[\sigma(X_{t \wedge \tau_2}^\varepsilon(x)) - \sigma(X_n^\varepsilon(\phi_n(t), x))]. \quad (3.10)$$

Then

$$|\tilde{b}_t| \leq c[\delta_1 + |m_t|], \quad (3.11)$$

$$\|\tilde{\sigma}_t\|^2 \leq c\varepsilon[\delta_1^2 + |m_t|^2], \quad (3.12)$$

$$\begin{aligned} m_t &= \int_0^{t \wedge \tau_2} \tilde{b}_s ds + \int_0^{t \wedge \tau_2} \tilde{\sigma}_s dB_s \\ &\quad - \int_0^{t \wedge \tau_2} \xi(X_{s \wedge \tau_2}^\varepsilon(x)) d|L^\varepsilon(x)|_s \\ &\quad + \int_0^{t \wedge \tau_2} \xi(X_n^\varepsilon(s \wedge \tau_2, x)) d|L_n^\varepsilon(x)|_s. \end{aligned} \quad (3.13)$$

Define

$$\begin{aligned} D_t &= \phi(X_{t \wedge \tau_2}^\varepsilon(x)) + \phi(X_n^\varepsilon(t \wedge \tau_2, x)), \\ N_t &= \exp\left\{-\frac{2}{\alpha} D_t\right\}. \end{aligned}$$

By Itô's formula

$$\begin{aligned} \Phi_\lambda(f(m_t)) &= 1 + \int_0^{t \wedge \tau_2} \Phi'_\lambda(f(m_s)) df(m_s) \\ &\quad + \frac{1}{2} \int_0^{t \wedge \tau_2} \Phi''_\lambda(f(m_s)) df(m_s) \cdot df(m_s), \end{aligned} \quad (3.14)$$

$$\begin{aligned} f(m_t)N_t &= \int_0^{t \wedge \tau_2} N_s df(m_s) + \int_0^{t \wedge \tau_2} f(m_s) dN_s \\ &\quad + \int_0^{t \wedge \tau_2} df(m_s) \cdot dN_s, \end{aligned} \quad (3.15)$$

$$\begin{aligned} f(m_t) &= 2 \int_0^{t \wedge \tau_2} m_s \cdot \tilde{b}_s ds + 2 \int_0^{t \wedge \tau_2} m_s \cdot \tilde{\sigma}_s dB_s \\ &\quad - 2 \int_0^{t \wedge \tau_2} m_s \cdot \xi(X_{s \wedge \tau_2}^\varepsilon(x)) d|L^\varepsilon(x)|_s \\ &\quad + 2 \int_0^{t \wedge \tau_2} m_s \cdot \xi(X_n^\varepsilon(s \wedge \tau_2, x)) d|L_n^\varepsilon(x)|_s \\ &\quad + \int_0^{t \wedge \tau_2} \text{trace}\{\tilde{\sigma}_s \tilde{\sigma}_s^T\} ds, \end{aligned} \quad (3.16)$$

$$dN_t = -\frac{2}{\alpha}N_t dD_t + \frac{2}{\alpha^2}N_t dD_t \cdot dD_t, \quad (3.17)$$

$$\begin{aligned} dD_t = & \left[(\nabla\phi^T \cdot b)(X_{t \wedge \tau_2}^\varepsilon(x)) + (\nabla\phi^T \cdot b)(X_n^\varepsilon(t \wedge \tau_2, x)) \right] dt \\ & - \left[(\nabla\phi^T \cdot \xi)(X_{t \wedge \tau_2}^\varepsilon(x)) d|L^\varepsilon(x)|_t \right. \\ & \left. + (\nabla\phi^T \cdot b)(X_n^\varepsilon(t \wedge \tau_2, x)) d|L_n^\varepsilon(x)|_t \right] \\ & + \sqrt{\varepsilon} \left[(\nabla\phi^T \cdot \sigma)(X_{t \wedge \tau_2}^\varepsilon(x)) + (\nabla\phi^T \cdot \sigma)(X_n^\varepsilon(t \wedge \tau_2, x)) \right] dB_t \\ & + \frac{\varepsilon}{2} \mathbf{trace} \{ (\nabla^2\phi \cdot \sigma \cdot \sigma^T)(X_{t \wedge \tau_2}^\varepsilon(x)) \\ & + (\nabla^2\phi \cdot \sigma \cdot \sigma^T)(X_n^\varepsilon(t \wedge \tau_2, x)) \} dt. \end{aligned} \quad (3.18)$$

Therefore, we have the following stochastic contractions:

$$\begin{aligned} dD_t \cdot dD_t = & \varepsilon \mathbf{trace} \{ (\nabla\phi^T \cdot \sigma \cdot \nabla\phi \cdot \sigma^T)(X_{t \wedge \tau_2}^\varepsilon(x)) \\ & (\nabla\phi^T \cdot \sigma \cdot \nabla\phi \cdot \sigma^T)(X_n^\varepsilon(t \wedge \tau_2, x)) \} dt, \end{aligned} \quad (3.19)$$

$$df(m_t) \cdot df(m_t) = 4 \mathbf{trace} \{ m_t \cdot \tilde{\sigma}_t \cdot \tilde{\sigma}_t^T m_t^T \} dt. \quad (3.20)$$

By using (3.17)-(3.19),

$$\begin{aligned} dN_t = & -\frac{2}{\alpha}N_t \sqrt{\varepsilon} \left[(\nabla\phi^T \cdot \sigma)(X_{t \wedge \tau_2}^\varepsilon(x)) + (\nabla\phi^T \cdot \sigma)(X_n^\varepsilon(t \wedge \tau_2, x)) \right] dB_t \\ & + \frac{2}{\alpha}N_t \left[(\nabla\phi^T \cdot \xi)(X_{t \wedge \tau_2}^\varepsilon(x)) d|L^\varepsilon(x)|_t \right. \\ & \left. + (\nabla\phi^T \cdot \xi)(X_n^\varepsilon(t \wedge \tau_2, x)) d|L_n^\varepsilon(x)|_t \right] \\ & - \frac{2}{\alpha}N_t \left[(\nabla\phi^T \cdot b)(X_{t \wedge \tau_2}^\varepsilon(x)) + (\nabla\phi^T \cdot b)(X_n^\varepsilon(t \wedge \tau_2, x)) \right] dt \\ & - \frac{\varepsilon}{\alpha}N_t \mathbf{trace} \{ (\nabla^2\phi \cdot \sigma \cdot \sigma^T)(X_{t \wedge \tau_2}^\varepsilon(x)) \\ & + (\nabla^2\phi \cdot \sigma \cdot \sigma^T)(X_n^\varepsilon(t \wedge \tau_2, x)) \} dt \\ & + \frac{2}{\alpha^2}N_t \varepsilon \mathbf{trace} \{ (\nabla\phi^T \cdot \sigma \cdot \nabla\phi \cdot \sigma^T)(X_{t \wedge \tau_2}^\varepsilon(x)) \\ & + (\nabla\phi^T \cdot \sigma \cdot \nabla\phi \cdot \sigma^T)(X_n^\varepsilon(t \wedge \tau_2, x)) \} dt. \end{aligned} \quad (3.21)$$

So the stochastic contraction $df(m_t) \cdot dN_t$ can be given by

$$\begin{aligned} df(m_t) \cdot dN_t = & -\frac{4}{\alpha}N_t \sqrt{\varepsilon} \mathbf{trace} \{ \tilde{\sigma}_t^T m_t^T \left[(\nabla\phi^T \cdot \sigma)(X_{t \wedge \tau_2}^\varepsilon(x)) \right. \\ & \left. + (\nabla\phi^T \cdot \sigma)(X_n^\varepsilon(t \wedge \tau_2, x)) \right] \} dt. \end{aligned} \quad (3.22)$$

Using (3.15), (3.16), (3.21) and (3.22) we obtain that

$$\begin{aligned}
f(m_t)N_t &= 2 \int_0^{t \wedge \tau_2} N_s \left\{ m_s \cdot \tilde{\sigma}_s - \frac{\sqrt{\varepsilon}}{\alpha} f(m_s) [(\nabla \phi^T \cdot \sigma)(X_{s \wedge \tau_2}^\varepsilon(x)) \right. \\
&\quad \left. + (\nabla \phi^T \cdot \sigma)(X_n^\varepsilon(s \wedge \tau_2, x))] \right\} dB_s \\
&\quad + 2 \int_0^{t \wedge \tau_2} N_s \left[\frac{1}{\alpha} |m_s|^2 (\nabla \phi(X_{s \wedge \tau_2}^\varepsilon(x)), \xi(X_{s \wedge \tau_2}^\varepsilon(x))) \right. \\
&\quad \left. - (m_s, \xi(X_{s \wedge \tau_2}^\varepsilon(x))) \right] d|L^\varepsilon(x)|_s \\
&\quad + 2 \int_0^{t \wedge \tau_2} N_s \left[\frac{1}{\alpha} |m_s|^2 (\nabla \phi(X_n^\varepsilon(s \wedge \tau_2, x)), \xi(X_n^\varepsilon(s \wedge \tau_2, x))) \right. \\
&\quad \left. - (m_s, \xi(X_n^\varepsilon(s \wedge \tau_2, x))) \right] d|L_n^\varepsilon(x)|_s \\
&\quad + \int_0^{t \wedge \tau_2} N_s [2m_s \cdot \tilde{b}_s + \mathbf{trace}\{\tilde{\sigma}_s \tilde{\sigma}_s^T\}] ds \\
&\quad - \int_0^{t \wedge \tau_2} \frac{2}{\alpha} N_s f(m_s) [(\nabla \phi^T \cdot b)(X_{s \wedge \tau_2}^\varepsilon(x)) \\
&\quad \quad + (\nabla \phi^T \cdot b)(X_n^\varepsilon(s \wedge \tau_2, x))] ds \\
&\quad - \int_0^{t \wedge \tau_2} \frac{\varepsilon}{\alpha} N_s f(m_s) \mathbf{trace}\{(\nabla^2 \phi \cdot \sigma \cdot \sigma^T)(X_{s \wedge \tau_2}^\varepsilon(x)) \\
&\quad \quad + (\nabla^2 \phi \cdot \sigma \cdot \sigma^T)(X_n^\varepsilon(s \wedge \tau_2, x))\} ds \\
&\quad + \int_0^{t \wedge \tau_2} \frac{2\varepsilon}{\alpha^2} N_s \mathbf{trace}\{(\nabla \phi^T \cdot \sigma \cdot \nabla \phi \cdot \sigma^T)(X_{s \wedge \tau_2}^\varepsilon(x)) \\
&\quad \quad + (\nabla \phi^T \cdot \sigma \cdot \nabla \phi \cdot \sigma^T)(X_n^\varepsilon(s \wedge \tau_2, x))\} ds, \\
&\quad - \int_0^{t \wedge \tau_2} \frac{4}{\alpha} N_s \sqrt{\varepsilon} \mathbf{trace}\{\tilde{\sigma}_s^T m_s^T [(\nabla \phi^T \cdot \sigma)(X_{s \wedge \tau_2}^\varepsilon(x)) \\
&\quad \quad + (\nabla \phi^T \cdot \sigma)(X_n^\varepsilon(s \wedge \tau_2, x))]\} ds \\
&:= \sum_{i=1}^8 a_i(t). \tag{3.23}
\end{aligned}$$

Using conditions (1.1)-(1.3) and (1.9), and $\phi \in C_b(\mathbb{R}^d)$, \exists a positive constant c such that $\frac{1}{c} \leq N_t \leq c$, since $a_2(t) + a_3(t) \leq 0$, by (3.23), we have

$$f(m_t) \leq c \{a_1(t) + \sum_{i=4}^8 a_i(t)\}. \tag{3.24}$$

By (3.14), (3.20) and (3.24)

$$\begin{aligned}
\Phi_\lambda(f(m_t)) &\leq 1 + c \int_0^{t \wedge \tau_2} \Phi'_\lambda(f(m_s)) da_1(s) \\
&+ c \sum_{i=4}^8 \int_0^{t \wedge \tau_2} \Phi'_\lambda(f(m_s)) da_i(s) \\
&+ 2 \int_0^{t \wedge \tau_2} \Phi''_\lambda(f(m_s)) \mathbf{trace}\{m_s \cdot \tilde{\sigma}_s \cdot \tilde{\sigma}_s^T m_s^T\} ds.
\end{aligned} \tag{3.25}$$

So, taking mathematical expectation at both sides of (3.25), we have

$$\begin{aligned}
\mathbf{E}\{\Phi_\lambda(f(m_t))\} &\leq 1 + c \sum_{i=4}^8 \mathbf{E}\left\{ \int_0^{t \wedge \tau_2} \Phi'_\lambda(f(m_s)) da_i(s) \right\} \\
&+ 2 \mathbf{E}\left\{ \int_0^{t \wedge \tau_2} \Phi''_\lambda(f(m_s)) \mathbf{trace}\{m_s \cdot \tilde{\sigma}_s \cdot \tilde{\sigma}_s^T m_s^T\} ds \right\} \\
&:= 1 + J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) + J_6(t).
\end{aligned} \tag{3.26}$$

Now we estimate the terms $J_i(t)$, $i = 1, \dots, 6$.

By (3.11), (3.12) and N_t is bounded we get that

$$\begin{aligned}
J_1(t) &= c \mathbf{E}\left\{ \int_0^{t \wedge \tau_2} \Phi'_\lambda(f(m_s)) N_s [2m_s \cdot \tilde{b}_s + \mathbf{trace}\{\tilde{\sigma}_s \tilde{\sigma}_s^T\}] ds \right\} \\
&\leq c^2 \mathbf{E}\left\{ \int_0^{t \wedge \tau_2} \frac{\lambda \Phi_\lambda(f(m_s))}{\delta_1^2 + |m_s|^2} [(3 + \varepsilon^2)(\delta_1^2 + |m_s|^2)] ds \right\} \\
&\leq \lambda(3 + \varepsilon) c^2 \mathbf{E}\left\{ \int_0^{t \wedge \tau_2} \Phi_\lambda(f(m_s)) ds \right\}.
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
|J_2(t)| &= \left| -\frac{2c}{\alpha} \mathbf{E}\left\{ \int_0^{t \wedge \tau_2} \Phi'_\lambda(f(m_s)) f(m_s) N_s [(\nabla \phi^T \cdot b)(X_{s \wedge \tau_2}^\varepsilon(x)) \right. \right. \\
&\quad \left. \left. + (\nabla \phi^T \cdot b)(X_n^\varepsilon(s \wedge \tau_2, x))] \right\} ds \right| \\
&\leq \lambda c \mathbf{E}\left\{ \int_0^{t \wedge \tau_2} \frac{\Phi_\lambda(f(m_s))}{\delta_1^2 + |m_s|^2} |m_s|^2 ds \right\} \\
&\leq \lambda c \mathbf{E}\left\{ \int_0^{t \wedge \tau_2} \Phi_\lambda(f(m_s)) ds \right\}
\end{aligned} \tag{3.28}$$

because $\nabla \phi$, b and N are bounded.

Similarly,

$$\begin{aligned}
|J_3(t)| &= \left| \frac{\varepsilon}{\alpha} \mathbf{E} \left\{ \int_0^{t \wedge \tau_2} \Phi'_\lambda(f(m_s)) N_s \mathbf{trace} \left\{ (\nabla^2 \phi \cdot \sigma \cdot \sigma^T)(X_{s \wedge \tau_2}^\varepsilon(x)) \right. \right. \right. \\
&\quad \left. \left. \left. + (\nabla^2 \phi \cdot \sigma \cdot \sigma^T)(X_n^\varepsilon(s \wedge \tau_2, x)) \right\} ds \right\} \right| \\
&\leq \lambda c \varepsilon \mathbf{E} \left\{ \int_0^{t \wedge \tau_2} \Phi_\lambda(f(m_s)) ds \right\}. \tag{3.29}
\end{aligned}$$

$$\begin{aligned}
|J_4(t)| &= \frac{2\varepsilon}{\alpha^2} \left| \mathbf{E} \left\{ \int_0^{t \wedge \tau_2} \Phi'_\lambda(f(m_s)) N_s \mathbf{trace} \left\{ (\nabla \phi^T \cdot \sigma \cdot \nabla \phi \cdot \sigma^T)(X_{s \wedge \tau_2}^\varepsilon(x)) \right. \right. \right. \\
&\quad \left. \left. \left. + (\nabla \phi^T \cdot \sigma \cdot \nabla \phi \cdot \sigma^T)(X_n^\varepsilon(s \wedge \tau_2, x)) \right\} ds \right\} \right| \\
&\leq \lambda c \varepsilon \mathbf{E} \left\{ \int_0^{t \wedge \tau_2} \Phi_\lambda(f(m_s)) ds \right\}. \tag{3.30}
\end{aligned}$$

Using (3.12), and σ and $\nabla \phi$ are bounded we have

$$\begin{aligned}
|J_5(t)| &= \frac{4\sqrt{\varepsilon}}{\alpha} \left| \mathbf{E} \left\{ \int_0^{t \wedge \tau_2} \Phi'_\lambda(f(m_s)) N_s \mathbf{trace} \left\{ \tilde{\sigma}_s^T m_s^T [(\nabla \phi^T \cdot \sigma)(X_{s \wedge \tau_2}^\varepsilon(x)) \right. \right. \right. \\
&\quad \left. \left. \left. + (\nabla \phi^T \cdot \sigma)(X_n^\varepsilon(s \wedge \tau_2, x))] \right\} ds \right\} \right| \\
&\leq \lambda c \varepsilon \mathbf{E} \left\{ \int_0^{t \wedge \tau_2} \frac{\Phi_\lambda(f(m_s))}{\delta_1^2 + |m_s|^2} \sqrt{\delta_1^2 + |m_s|^2} |m_s| ds \right\} \\
&\leq \lambda c \varepsilon \mathbf{E} \left\{ \int_0^{t \wedge \tau_2} \Phi_\lambda(f(m_s)) ds \right\}. \tag{3.31}
\end{aligned}$$

Since $\Phi''_\lambda(x) = \frac{\lambda(\lambda-1)\Phi_\lambda(x)}{(\delta_1^2+x)^2}$, $\forall x \geq 0$, by (3.12), we get that

$$\begin{aligned}
|J_6(t)| &= \left| 2 \mathbf{E} \left\{ \int_0^{t \wedge \tau_2} \Phi''_\lambda(f(m_s)) \mathbf{trace} \{ m_s \cdot \tilde{\sigma}_s \cdot \tilde{\sigma}_s^T m_s^T \} ds \right\} \right| \\
&\leq \lambda(\lambda-1) c \varepsilon \mathbf{E} \left\{ \int_0^{t \wedge \tau_2} \frac{\Phi_\lambda(f(m_s))}{(\delta_1^2 + |m_s|^2)^2} (\delta_1^2 + |m_s|^2) |m_s|^2 ds \right\} \\
&\leq \lambda(\lambda-1) c \varepsilon \mathbf{E} \left\{ \int_0^{t \wedge \tau_2} \Phi_\lambda(f(m_s)) ds \right\}. \tag{3.32}
\end{aligned}$$

Therefore, putting (3.27)-(3.32) and (3.26) together, we have

$$\mathbf{E} \{ \Phi_\lambda(f(m_t)) \} \leq 1 + c[\lambda + \lambda\varepsilon + \lambda^2\varepsilon] \int_0^t \mathbf{E} \{ \Phi_\lambda(f(m_{s \wedge \tau_2})) \} ds,$$

which, by Gronwall lemma, implies that

$$\mathbf{E} \{ \Phi_\lambda(f(m_t)) \} \leq \exp \{ c[\lambda + \lambda\varepsilon + \lambda^2\varepsilon]t \}. \tag{3.33}$$

Letting $t = 1$ in the last inequality we have

$$\mathbf{P} \{ \tau_1 \geq 1, \tau \leq 1 \} \Phi_\lambda(\delta^2) \leq \exp \{ c[\lambda + \lambda\varepsilon + \lambda^2\varepsilon] \}.$$

So

$$\varepsilon \log \mathbf{P}\{\tau_1 \geq 1, \tau \leq 1\} \leq c[\lambda\varepsilon + \lambda\varepsilon^2 + \lambda^2\varepsilon^2] - \lambda\varepsilon \int_0^{\delta^2} \frac{ds}{\delta_1^2 + s}. \quad (3.34)$$

Taking $\lambda\varepsilon = 1$, then making ε tend to $+\infty$, finally letting δ_1 tend to 0 in (3.34), we have

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\{\tau_1 \geq 1, \tau \leq 1\} = -\infty. \quad (3.35)$$

Since

$$\mathbf{P}\left\{\sup_{t \in [0,1]} |X_t^\varepsilon(x) - X_n^\varepsilon(t, x)| \geq \delta\right\} \leq \mathbf{P}\{\tau_1 \leq 1\} + \mathbf{P}\{\tau_1 \geq 1, \tau_2 \leq 1\},$$

we immediately deduce from (3.7) and (3.35) that

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\left\{\sup_{t \in [0,1]} |X_t^\varepsilon(x) - X_n^\varepsilon(t, x)| \geq \delta\right\} = -\infty \quad (3.36)$$

for any $\delta > 0$. Therefore, we complete the proof of Theorem 3.1. \square

4 Large deviations principles on SDE(1.4) with reflection

The main purpose of this section is to Theorem 1.1.

Proof of Theorem 1.1. By (2.22) and (3.1), for $n \geq 1$ we have

$$F_n(\sqrt{\varepsilon}B) = X_n^\varepsilon.$$

Using the Schilder theorem (see Theorem 1.3.27 in [4]), $\{\sqrt{\varepsilon}B, \varepsilon > 0\}$ satisfies the large deviations principles on E_1 with good rate function $I_1(\cdot)$ defined by (1.6). Because of Theorem 2.3 $F_n(\cdot)$ is a continuous map from E_1 to E , therefore, by the contraction principle (see Theorem 4.2.1 [3]), $\{X_n^\varepsilon, \varepsilon > 0\}$ satisfies the large deviations principles on E with good rate function $I_{2,n}^x$ defined by

$$I_{2,n}^x(f) = \inf \left\{ I_1(\psi) : f = z_n^\psi \text{ and } \psi \in L^2([0, 1], \mathbb{R}^d) \right\}$$

for $f \in E$, where z_n^ψ is solution of Eq.(2.14). By Theorem 2.2 and Theorem 3.1 as well as the generalized contraction principle for large deviations principles (see Theorem 4.2.23 in [3]), the family $\{X^\varepsilon, \varepsilon > 0\}$ satisfies the large deviations principles on E with the rate function $I_2^x(\cdot)$ defined by (1.8). So we complete the proof. \square

Using the same way as in Theorem 1.1, we can prove the following and we omit its proof here.

Theorem 4.1. *Assume the conditions of Theorem 1.1. $\{X^\varepsilon(t, y)\}$ is solution of SDE(1.4) corresponding to the initial condition $X_0 = y$. Then for any open set G and any closed set F of E we have*

$$\liminf_{\substack{\varepsilon \rightarrow 0 \\ y \rightarrow x}} \varepsilon \log \mathbf{P}\{X^\varepsilon(y) \in G\} \geq - \inf_{f \in G} \{I_2^x(f)\}, \quad (4.1)$$

$$\limsup_{\substack{\varepsilon \rightarrow 0 \\ y \rightarrow x}} \varepsilon \log \mathbf{P}\{X^\varepsilon(y) \in F\} \leq - \inf_{f \in F} \{I_2^x(f)\}. \quad (4.2)$$

As a direct consequence of Theorem 4.1 we also have the following.

Proposition 4.1. *Assume the conditions of Theorem 1.1. Then for any compact set $K \subset \bar{O}$, any open set G and any closed set F of E we have*

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \inf_{y \in K} \mathbf{P}\{X^\varepsilon(y) \in G\} \geq - \sup_{y \in K} \inf_{f \in G} \{I_2^y(f)\}, \quad (4.3)$$

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \sup_{y \in K} \mathbf{P}\{X^\varepsilon(y) \in F\} \leq - \inf_{\substack{f \in G \\ y \in K}} \{I_2^y(f)\}. \quad (4.4)$$

5 Large deviations principles on SDE(1.4) with reflection and anticipated initial conditions

Let X_0^ε be any random variable, which may not be adapted to \mathcal{F}_0 , $X_t^\varepsilon(x)$ be a solution of SDE(1.4) corresponding to initial data x . The author proved in [6] that the composition $X_t^\varepsilon(X_0^\varepsilon)$ is also a solution of SDE (1.4) corresponding to anticipated initial conditions X_0^ε . Therefore the integral in this equation should be anticipated in Malliavin sense (see [9]). The main purpose of this section is to prove Theorem 1.2 via Theorem 1.1 and Proposition 4.1. The proof is the following.

Proof of Theorem 1.2. *We first prove that the large deviations lower bound holds for $\{X_t^\varepsilon(X_0^\varepsilon), \varepsilon > 0\}$ with rate function $I^{x_0}(\cdot)$ on $E = C([0, 1], \bar{O})$ if the conditions of Theorem 1.2 are satisfied.*

Let $G \subset E$ be an open set. Assume that $g \in G$ with $I^{x_0}(g) < +\infty$. By the definition of $I^{x_0}(g)$, there exist $\delta_2 > 0$, $\psi \in L^2([0, 1]; \mathbb{R}^d)$ and $z^\psi(y)$, a solution of Eq.(1.5) corresponding to initial y , such that $|y - x_0| < \delta_2$ and

$$I_2^y(g) = I_2^y(z^\psi(y)) = I_1(\psi) \leq I^{x_0}(g) < +\infty. \quad (5.1)$$

Using the same way as in (2.8), we can prove that for any $y_1, y_2 \in \mathbb{R}^d$

$$\sup_{t \in [0, 1]} \{|z^\psi(y_1) - z^\psi(y_2)|^2\} \leq c|y_1 - y_2|^2 \exp\{c(1 + \|\psi\|_{L_2})\} \quad (5.2)$$

for some constants $c = c(\alpha, C_0, O)$. Taking $\delta_3 > 0$ such that

$$B_{\delta_3}(y) = \{f : \|f - z^\psi(y)\|_E \leq \delta_3\}, \quad (5.3)$$

we have

$$\begin{aligned}
\mathbf{P}\{X^\varepsilon(X_0^\varepsilon) \in G\} &\geq \mathbf{P}\{\|X^\varepsilon(X_0^\varepsilon) - z^\psi(y)\|_E \leq \delta_3, |X_0^\varepsilon - y| \leq 2\delta_3\} \\
&\geq \mathbf{P}\left\{\sup_{|x-y| \leq 2\delta_3} \|X^\varepsilon(x) - z^\psi(y)\|_E \leq \delta_3, |X_0^\varepsilon - y| \leq 2\delta_3\right\} \\
&\quad (\text{by (5.2)}) \\
&\geq \mathbf{P}\left\{\sup_{|x-y| \leq 2\delta_3} \|X^\varepsilon(x) - z^\psi(x)\|_E \leq \frac{\delta_3}{2}, |X_0^\varepsilon - y| \leq 2\delta_3\right\} \\
&= \mathbf{P}\left\{\sup_{|x-y| \leq 2\delta_3} \|X^\varepsilon(x) - z^\psi(x)\|_E \leq \frac{\delta_3}{2}\right\} \\
&\quad - \mathbf{P}\{|X_0^\varepsilon - y| > 2\delta_3\} \\
&\geq \inf_{|x-y| \leq 2\delta_3} \mathbf{P}\left\{\|X^\varepsilon(x) - z^\psi(x)\|_E < \frac{\delta_3}{2}\right\} \\
&\quad - \mathbf{P}\{|X_0^\varepsilon - y| > 2\delta_3\}. \tag{5.4}
\end{aligned}$$

Using (4.3), (5.1) and Theorem 1.1 we have

$$\begin{aligned}
&\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \inf_{|x-y| \leq 2\delta_3} \mathbf{P}\left\{\|X^\varepsilon(x) - z^\psi(x)\|_E < \frac{\delta_3}{2}\right\} \\
&\geq \inf_{|x-y| \leq 2\delta_3} \left[- \inf_{f \in B_{\frac{\delta_3}{2}}(z^\psi(y))} \{I_2^y(f)\} \right] \\
&\geq \inf_{|x-y| \leq 2\delta_3} \left[- I_2^y(z^\psi(y)) \right] \\
&\geq \inf_{|x-y| \leq 2\delta_3} \left[- I_1(\psi) \right] = -I_1(\psi) \geq -I^{x_0}(g). \tag{5.5}
\end{aligned}$$

Therefore, putting (1.14), (5.4) and (5.5) together, we have

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\{X^\varepsilon(X_0^\varepsilon) \in G\} \geq -I^{x_0}(g).$$

Since $g \in G$ is arbitrary, we have

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\{X^\varepsilon(X_0^\varepsilon) \in G\} \geq - \inf_{f \in G} \{I^{x_0}(g)\}.$$

So we complete the proof of the large deviations lower bound.

Next we prove that the large deviations upper bound holds for $\{X_t^\varepsilon(X_0^\varepsilon), \varepsilon > 0\}$ with rate function $I^{x_0}(\cdot)$ on E if the conditions of Theorem 1.2 are satisfied.

Let $F \subset E$ be a closed set. If $\inf_{f \in F} \{I^{x_0}(f)\} = 0$, then (1.16) is trivial. We need only to prove that for any $a > 0$ with $a < \inf_{f \in F} \{I^{x_0}(f)\}$

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\{X^\varepsilon(X_0^\varepsilon) \in F\} \leq -a. \tag{5.6}$$

Now we assume that $\exists a > 0$ such that

$$\inf_{f \in F} \{I^{x_0}(f)\} > a.$$

Then there exists $\delta_4 > 0$ and $y_0 \in B(x_0, \delta_4) = \{x : |x - x_0| \leq \delta_4\}$ such that

$$\inf_{f \in F} \{I_2^{y_0}(f)\} > a. \quad (5.7)$$

Then the level set $K_a(y_0) := \{f : I_2^{y_0}(f) \leq a\} = \{z^\psi(y_0) : I_2^{y_0}(z^\psi(y_0)) \leq a, \psi \in L^2([0, 1]; \mathbb{R}^d)\}$, $z^\psi(y_0)$ is a solution of Eq.(1.5) corresponding to initial data y_0 . Since $K_a(y_0) \cap F = \emptyset$, $\forall f = z^\psi(y_0) \in K_a(y_0)$, $\exists \delta_f > 0$ such that

$$U_{\delta_f}(f) = \{g : \|g - f\|_{E_2} < \delta_f\} \cap F = \emptyset.$$

Noting that $\bigcup_{f \in K_a(y_0)} U_{\delta_f}(f) \supset K_a(y_0)$ and $K_a(y_0)$ is compact, $\exists m > 0$ and $f_j \in K_a(y_0)$ such that

$$\mathbf{U} := \bigcup_{j=1}^m U_{\delta_{f_j}}(f_j) \supset K_a(y_0). \quad (5.8)$$

So

$$\mathbf{U} \cap F = \emptyset. \quad (5.9)$$

Hence

$$\begin{aligned} & \mathbf{P}\{X^\varepsilon(X_0^\varepsilon) \in F\} \\ & \leq \mathbf{P}\{X^\varepsilon(X_0^\varepsilon) \in F, |X^\varepsilon - y_0| \leq 2\delta_4\} + \mathbf{P}\{|X_0^\varepsilon - x_0| \geq \delta_4\} \\ & := \square_1 + \square_2. \end{aligned} \quad (5.10)$$

By (5.9)

$$\begin{aligned} \square_1 & \leq \sup_{|y-y_0| \leq 2\delta_4} \mathbf{P}\{X^\varepsilon(y) \in F\} \\ & \leq \sup_{|y-y_0| \leq 2\delta_4} \mathbf{P}\{X^\varepsilon(y) \in \mathbf{U}^c\} \\ & \leq \sup_{|y-y_0| \leq 2\delta_4} \mathbf{P}\{X^\varepsilon(y_0) \in \mathbf{U}^c\} \\ & \quad + \sup_{|y-y_0| \leq 2\delta_4} \limsup_{k \rightarrow \infty} \mathbf{P}\{|X^\varepsilon(y) - X^\varepsilon(y_0)| \geq \frac{1}{k}\} \\ & \leq \sup_{|y-y_0| \leq 2\delta_4} \mathbf{P}\{X^\varepsilon(y_0) \in \mathbf{U}^c\} \\ & \quad + \limsup_{k \rightarrow \infty} \sup_{|y-y_0| \leq 2\delta_4} \mathbf{P}\{|X^\varepsilon(y) - X^\varepsilon(y_0)| \geq \frac{1}{k}\} \\ & := \square_{11} + \limsup_{k \rightarrow \infty} \square_{12}. \end{aligned} \quad (5.11)$$

Using Theorem 1.1 and \mathbf{U}^c is a closed set, we have

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \square_{11} \leq - \inf_{f \in \mathbf{U}^c} \{I_2^{y_0}(f)\} \leq -a \quad (5.12)$$

because $\mathbf{U}^c \subset K_a(y_0)^c$.

We claim that

$$\limsup_{k \rightarrow \infty} \limsup_{\delta_4 \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \square_{12} = -\infty. \quad (5.13)$$

To prove (5.13) we define for $k \geq 1$

$$\tau = \inf\{t : |X_t^\varepsilon(y) - X_t^\varepsilon(y_0)| \geq \frac{1}{k}\} \wedge 1, \quad (5.14)$$

$$m_t = X_{t \wedge \tau}^\varepsilon(y) - X_{t \wedge \tau}^\varepsilon(y_0), \quad (5.15)$$

$$b_1(t) = b(X_{t \wedge \tau}^\varepsilon(y)) - b(X_{t \wedge \tau}^\varepsilon(y_0)), \quad (5.16)$$

$$\sigma_1(t) = \sqrt{\varepsilon}[\sigma(X_{t \wedge \tau}^\varepsilon(y)) - \sigma(X_{t \wedge \tau}^\varepsilon(y_0))]. \quad (5.17)$$

Then

$$\begin{aligned} m_t &= y - y_0 + \int_0^t b_1(s)ds + \int_0^t b_1(s)dB(s) \\ &\quad - [L_{t \wedge \tau}^\varepsilon(y) - L_{t \wedge \tau}^\varepsilon(y_0)]. \end{aligned} \quad (5.18)$$

Define functions Ψ and Φ_λ on $[0, +\infty)$ by

$$\Psi(x) = \int_0^x \frac{ds}{\rho^2 + s} \quad \text{and} \quad \Phi_\lambda(x) = \exp\{\lambda \Psi(x)\} \quad \text{for } \lambda > 0 \text{ and } \rho > 0.$$

By the same way as in (3.36), we obtain that

$$\limsup_{\delta_4 \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \square_{12} \leq - \int_0^{\frac{1}{k^2}} \frac{ds}{\rho^2 + s} + c \quad (5.19)$$

for some constants c and any $k \geq 1$. Letting $\rho \rightarrow 0$ we have for any $k \geq 1$

$$\limsup_{\delta_4 \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \square_{12} = -\infty. \quad (5.20)$$

This implies claim (5.13).

Therefore, we deduce from that (5.11) to (5.13) that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \square_1 &\leq \left\{ \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \square_{11} \right\} \\ &\quad \vee \left\{ \limsup_{k \rightarrow \infty} \limsup_{\delta_4 \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \square_{12} \right\} \\ &\leq -a. \end{aligned} \quad (5.21)$$

So we know from (1.13), (5.10) and (5.21) that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\{X^\varepsilon(X_0^\varepsilon) \in F\} &\leq \left\{ \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \square_1 \right\} \\ &\quad \vee \left\{ \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{P}\{|X_0^\varepsilon - x_0| > \delta_4\} \right\} \\ &\leq -a, \end{aligned} \quad (5.22)$$

that is, (5.6) holds. Thus we complete the proof of the large deviations upper bound.

Finally, we prove that $I^x(\cdot)$ is a good rate function on E .

Since for any $a \in [0, \infty)$

$$\{f : I^x(f) > a\} = \bigcup_{n=1} \bigcup_{y: |y-x_0| \leq \frac{1}{n}} \{f : I_2^y(f) > a\}$$

and $I_2^y(\cdot)$ is a good rate function, we know that $\{f : I^x(f) > a\}$ is an open set and $\{f : I_2^y(f) \leq a\}$ is compact set. So I^x is a lower semicontinuous function. Because $\{f : I^x(f) \leq a\} \subset \{f : I_2^y(f) \leq a\}$ for $y \in B(x_0, 1)$, $\{f : I^x(f) \leq a\}$ is also compact, so $I^x(\cdot)$ is also a good rate function on E . Thus we complete the proof of Theorem 1.2. \square .

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